

# CERTAIN MAPS PRESERVING SELF-HOMOTOPY EQUIVALENCES

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**ABSTRACT.** Let  $\mathcal{E}(X)$  be the group of homotopy classes of self homotopy equivalences for a connected CW complex  $X$ . We observe two classes of maps  $\mathcal{E}$ -maps and co- $\mathcal{E}$ -maps. They are defined as the maps  $X \rightarrow Y$  that induce the homomorphisms  $\mathcal{E}(X) \rightarrow \mathcal{E}(Y)$  and  $\mathcal{E}(Y) \rightarrow \mathcal{E}(X)$ , respectively. We give some rationalized examples related to spheres, Lie groups and homogeneous spaces by using Sullivan models. Furthermore, we introduce an  $\mathcal{E}$ -equivalence relation between rationalized spaces  $X_{\mathbb{Q}}$  and  $Y_{\mathbb{Q}}$  as a geometric realization of an isomorphism  $\mathcal{E}(X_{\mathbb{Q}}) \cong \mathcal{E}(Y_{\mathbb{Q}})$ .

## 1. INTRODUCTION

Needless to say, the based homotopy set  $[X, Y]$  of based continuous maps from a based space  $X$  to a based space  $Y$  is a most interesting object in homotopy theory. In the following, all maps are based and we do not distinguish a homotopy class and the representative in a homotopy set. Let  $X$  be a connected CW complex with base point  $*$  and let

$$\mathcal{E}(X) = \{[f] \in [X, X] \mid f : X \xrightarrow{\sim} X\}$$

be the group of homotopy classes of self-homotopy equivalences for  $X$  with the operation given by the composition of homotopy classes. This group is important and has been closely studied as part of homotopy theory (for example, see [3], [18], [19], [20], [6], [7], [8]).

It is clear that  $\mathcal{E}(X) \cong \mathcal{E}(Y)$  as a group if  $X \simeq Y$ . One of the difficulties of its computation or evaluation may be based on the fact that  $\mathcal{E}(\ )$  is not functorial, i.e., there is no suitable induced map between  $\mathcal{E}(X)$  and  $\mathcal{E}(Y)$  for the map  $f : X \rightarrow Y$  in general. However, recall that, for example, the injection  $i_X : X \rightarrow X \times Y$  and the projection  $p_Y : X \times Y \rightarrow Y$  induce the natural monomorphisms  $\mathcal{E}(X) \rightarrow \mathcal{E}(X \times Y)$  and  $\mathcal{E}(Y) \rightarrow \mathcal{E}(X \times Y)$ , respectively.

**Definition 1.1.** We say a map  $f : X \rightarrow Y$  is an  **$\mathcal{E}$ -map** if there is a homomorphism  $\phi_f : \mathcal{E}(X) \rightarrow \mathcal{E}(Y)$  such that

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ f \downarrow & & \downarrow f \\ Y & \xrightarrow{\phi_f(g)} & Y \end{array}$$

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homotopically commutes for any element  $g$  of  $\mathcal{E}(X)$ . We say the map  $f : X \rightarrow Y$  is a **co- $\mathcal{E}$ -map** if there is a homomorphism  $\psi_f : \mathcal{E}(Y) \rightarrow \mathcal{E}(X)$  such that

$$\begin{array}{ccc} X & \xrightarrow{\psi_f(g)} & X \\ f \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & Y \end{array}$$

homotopically commutes for any element  $g$  of  $\mathcal{E}(Y)$ .

Especially we consider the rationalized version of  $\mathcal{E}$ -maps and co- $\mathcal{E}$ -maps by using Sullivan models [10], [12], [21]. Let  $X_{\mathbb{Q}}$  be the rationalization of a nilpotent space  $X$  [15].

**Definition 1.2.** A map  $f : X \rightarrow Y$  between nilpotent spaces is a **rational  $\mathcal{E}$ -map** if the rationalization  $f_{\mathbb{Q}} : X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$  is an  $\mathcal{E}$ -map. Similarly a map  $f : X \rightarrow Y$  between nilpotent spaces is a **rational co- $\mathcal{E}$ -map** if the rationalization  $f_{\mathbb{Q}} : X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$  is a co- $\mathcal{E}$ -map.

**Question 1.3.** When is a map a (rational)  $\mathcal{E}$ -map or a (rational) co- $\mathcal{E}$ -map ?

**Theorem 1.4.** Let  $G$  be a compact connected Lie group and  $H$  be a connected closed sub-Lie group of  $G$ .

(1) The inclusion  $j : H \rightarrow G$  is a rational  $\mathcal{E}$ -map if  $\pi_*(j) \otimes \mathbb{Q}$  is injective.

(2) For the homogeneous space  $G/H$ , the projection map  $f : G \rightarrow G/H$  is a rational co- $\mathcal{E}$ -map.

The assumption  $\mathcal{E}(X) \cong \mathcal{E}(Y)$  does not, in general, imply  $X$  and  $Y$  are homotopy equivalent spaces. Finally, we consider the question: *When is an isomorphism  $\mathcal{E}(X_{\mathbb{Q}}) \cong \mathcal{E}(Y_{\mathbb{Q}})$  realized as a composition of rational  $\mathcal{E}$ -maps and rational co- $\mathcal{E}$ -maps between  $X_{\mathbb{Q}}$  and  $Y_{\mathbb{Q}}$ ?*

**Definition 1.5.** We say that nilpotent spaces  $X$  and  $Y$  are **rationally  $\mathcal{E}$ -equivalent** (denote as  $X_{\mathbb{Q}} \sim_{\mathcal{E}} Y_{\mathbb{Q}}$ ) if there is a chain of *spherically injective*  $\mathcal{E}$ -maps and *spherically injective* co- $\mathcal{E}$ -maps

$$X_{\mathbb{Q}} \xrightarrow{f_1} Z_1 \xleftarrow{f_2} \cdots \xleftarrow{f_n} Z_n \xrightarrow{f_{n+1}} Y_{\mathbb{Q}}$$

( $Z_i$  are rational spaces) such that an isomorphism  $\mathcal{E}(X_{\mathbb{Q}}) \cong \mathcal{E}(Y_{\mathbb{Q}})$  is given by a composition of  $n+1$ -isomorphisms  $\{\phi_{f_i}\}_i$  and  $\{\psi_{f_i}\}_i$ , i.e.,  $\phi_{f_{n+1}} \circ \psi_{f_n} \circ \cdots \circ \psi_{f_2} \circ \phi_{f_1} : \mathcal{E}(X_{\mathbb{Q}}) \xrightarrow{\cong} \mathcal{E}(Y_{\mathbb{Q}})$  or  $\psi_{f_1} \circ \phi_{f_2} \circ \cdots \circ \phi_{f_n} \circ \psi_{f_{n+1}} : \mathcal{E}(Y_{\mathbb{Q}}) \xrightarrow{\cong} \mathcal{E}(X_{\mathbb{Q}})$ .

**Remark 1.6.** In this paper, we say that a map  $f : X \rightarrow Y$  is *spherically injective* when  $f_{\sharp}(u) \neq 0 \in \pi_*(Y)$  if  $hur_X(u) \neq 0$  for  $u \in \pi_*(X)$ . Here  $hur_X : \pi_*(X) \rightarrow H_*(X)$  is the Hurewicz homomorphism for a space  $X$ . Thus we have

(weakly) homotopy equivalent  $\Rightarrow$  spherically injective  $\Rightarrow$  homotopy non-trivial

If we admit the homotopy trivial maps as  $f_i$ , any isomorphism  $\mathcal{E}(X_{\mathbb{Q}}) \cong \mathcal{E}(Y_{\mathbb{Q}})$  induces  $X_{\mathbb{Q}} \sim_{\mathcal{E}} Y_{\mathbb{Q}}$  by the constant map  $*$  :  $X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$ .

**Theorem 1.7.** (1) If  $X$  and  $Y$  are rationally homotopy equivalent, i.e.,  $X_{\mathbb{Q}} \simeq Y_{\mathbb{Q}}$ , then  $X_{\mathbb{Q}} \sim_{\mathcal{E}} Y_{\mathbb{Q}}$ .

(2) For any  $n$ ,  $S_{\mathbb{Q}}^2 \sim_{\mathcal{E}} \mathbb{C}P_{\mathbb{Q}}^n$  and  $S_{\mathbb{Q}}^4 \sim_{\mathcal{E}} \mathbb{H}P_{\mathbb{Q}}^n$ .

- (3) When  $m$  is even and  $n$  is odd,  $(S^m \vee S^n)_{\mathbb{Q}} \underset{\mathcal{E}}{\sim} (S^m \times S^n)_{\mathbb{Q}}$  if and only if  $m \neq n+1$ .  
(4) For odd-integers  $1 < m \leq n$ ,  $(S^m \times S^n)_{\mathbb{Q}} \underset{\mathcal{E}}{\sim} E_{\mathbb{Q}} \underset{\mathcal{E}}{\sim} E'_{\mathbb{Q}}$  for non-trivial fibrations  $S^{m+n-1} \rightarrow E \xrightarrow{p} S^m \times S^n$  and  $S^{2m+n-2} \rightarrow E' \xrightarrow{p'} E$ .  
(5) There are integers  $m \neq n$  such that  $S_{\mathbb{Q}}^m \underset{\mathcal{E}}{\sim} S_{\mathbb{Q}}^n$ . For example,  $S_{\mathbb{Q}}^{53} \underset{\mathcal{E}}{\sim} S_{\mathbb{Q}}^{67}$ .

**Remark 1.8.** The proof of Theorem 1.7 (5) requires a rigid rational space  $X$  of [4], which induces  $\mathcal{E}(X) = \{id_X\}$ . For the total space  $Z_1$  of a fibration

$$X \rightarrow Z_1 \rightarrow S_{\mathbb{Q}}^m \times S_{\mathbb{Q}}^n,$$

it is given by the sequence  $S_{\mathbb{Q}}^m \xrightarrow{f_1} Z_1 \xleftarrow{f_2} S_{\mathbb{Q}}^n$  of an  $\mathcal{E}$ -map  $f_1$  with  $\phi_{f_1} : \mathcal{E}(S_{\mathbb{Q}}^m) \cong \mathcal{E}(Z_1)$  and a co- $\mathcal{E}$ -map  $f_2$  with  $\psi_{f_2} : \mathcal{E}(Z_1) \cong \mathcal{E}(S_{\mathbb{Q}}^n)$ . Because of the demand that  $\mathcal{E}(Z_1) \cong \mathbb{Q}^*$  ( $\mathbb{Q}^* := \mathbb{Q} - 0$ ), we need suitable restrictions about the pair  $(m, n)$  and see that (53, 67) satisfies them in the proof. Of course, it depends on the model structure of  $X$ . So we may require more various types of rigid models for the proof of (5) in many cases of  $(m, n)$ . On the other hand, the authors cannot find an example that  $S_{\mathbb{Q}}^m \not\underset{\mathcal{E}}{\sim} S_{\mathbb{Q}}^n$  for some  $(m, n)$ .

**Problem 1.9.** If  $\mathcal{E}(X) \cong \mathcal{E}(Y)$  for rational spaces  $X$  and  $Y$ , does it hold that  $X \underset{\mathcal{E}}{\sim} Y$ ?

**Remark 1.10.** For rational spaces  $X$ ,  $Y$  and  $Z$ , even if  $Y \underset{\mathcal{E}}{\sim} Z$ , it may not hold that  $X \times Y \underset{\mathcal{E}}{\sim} X \times Z$ . For example, when  $X = S^5$ ,  $Y = S^2$  and  $Z = \mathbb{C}P^2$ ,  $\mathcal{E}((X \times Y)_{\mathbb{Q}}) \cong \mathbb{Q}^* \times \mathbb{Q}^*$  but  $\mathcal{E}((X \times Z)_{\mathbb{Q}})$  is isomorphic to the subgroup of lower triangular matrixes of  $GL(2, \mathbb{Q})$ .

In §2, we demonstrate the basic properties and provide examples in ordinary homotopy theory of  $\mathcal{E}$ -maps and co- $\mathcal{E}$ -maps. In §3, we give some computations in rational homotopy theory using Sullivan minimal models.

## 2. SOME PROPERTIES

Recall that  $[X, \ ]$  is the covariant functor from the category of spaces to the category of sets, where for a map  $f : Y \rightarrow Z$ , the map  $f_*(g) : [X, Y] \rightarrow [X, Z]$  is given by  $f_*(g) = f \circ g$ . On the other hand,  $[ \ , Z]$  is the contravariant functor. For the map  $f : X \rightarrow Y$ , the map  $f^*(g) : [Y, Z] \rightarrow [X, Z]$  is given by  $f^*(g) = g \circ f$ . The following lemma holds from  $\phi_f(g) \circ f = f \circ g$  and  $f \circ \psi_f(g) = g \circ f$ .

**Lemma 2.1.** A map  $f : X \rightarrow Y$  is an  $\mathcal{E}$ -map (or a co- $\mathcal{E}$ -map) if and only if there is a group homomorphism  $\phi_f : \mathcal{E}(X) \rightarrow \mathcal{E}(Y)$  (or  $\psi_f : \mathcal{E}(Y) \rightarrow \mathcal{E}(X)$ ) where the following diagrams

$$\begin{array}{ccc} [X, X] & \xrightarrow{f_*} [X, Y] & \xleftarrow{f^*} [Y, Y] \\ \uparrow \cup & & \uparrow \cup \\ \mathcal{E}(X) & \xrightarrow{\phi_f} & \mathcal{E}(Y) \end{array} \quad \begin{array}{ccc} [X, X] & \xrightarrow{f_*} [X, Y] & \xleftarrow{f^*} [Y, Y] \\ \uparrow \cup & & \uparrow \cup \\ \mathcal{E}(X) & \xleftarrow{\psi_f} & \mathcal{E}(Y) \end{array}$$

are commutative.

Of course, the maps  $\phi_f$  and  $\psi_f$  may not be uniquely determined for a map  $f$ .

**Lemma 2.2.** (1) If maps  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are  $\mathcal{E}$ -maps, then  $g \circ f : X \rightarrow Z$  is an  $\mathcal{E}$ -map.

(2) If  $f$  and  $g$  are co- $\mathcal{E}$ -maps, then  $g \circ f$  is a co- $\mathcal{E}$ -map.

(3) The constant map is both an  $\mathcal{E}$ -map and a co- $\mathcal{E}$ -map.

(4) A homotopy equivalence map is both an  $\mathcal{E}$ -map and a co- $\mathcal{E}$ -map.

*Proof.* (1)  $\phi_{g \circ f}(h) := \phi_g \circ \phi_f(h)$  for  $h \in \mathcal{E}(X)$ .

(2)  $\psi_{g \circ f}(h) := \psi_g \circ \psi_f(h)$  for  $h \in \mathcal{E}(Z)$ .

(3) It is sufficient to put  $\phi_f = \psi_f = *$ , i.e.,  $\phi_f(g) = id_Y$  and  $\psi_f(g) = id_X$  for any  $g$ .

(4) It is sufficient to put  $\phi_f(h) := f \circ h \circ f^{-1}$  for  $h \in \mathcal{E}(X)$  and  $\psi_f(h) := f^{-1} \circ h \circ f$  for  $h \in \mathcal{E}(Y)$ , where  $f^{-1}$  is the homotopy inverse of  $f$ .  $\square$

**Definition 2.3.** [14, Chapter 3]([16]) Let  $\alpha : X \rightarrow Y$  and  $\beta : Z \rightarrow W$  be maps.  $\Pi(\alpha, \beta)$  is the set of all homotopy classes of pairs  $[f_1, f_2]$  such that

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Z \\ \alpha \downarrow & & \downarrow \beta \\ Y & \xrightarrow{f_2} & W \end{array}$$

is homotopy commutative. Here a homotopy of  $(f_1, f_2)$  is just a pair of homotopies  $(f_{1t}, f_{2t})$  such that  $\beta f_{1t} = f_{2t} \alpha$ . If  $[f_1, f_2]$  has a two sided inverse in  $\Pi(\alpha, \beta)$ , we call  $[f_1, f_2]$  a homotopy equivalence. If  $\alpha = \beta$ , we call  $[f_1, f_2]$  a self-homotopy equivalence and denote the set of all self-homotopy equivalences by  $\mathcal{E}(\alpha)$ .

**Lemma 2.4.** For a map  $f : X \rightarrow Y$ ,

(1)  $f$  is an  $\mathcal{E}$ -map if and only if  $h : \mathcal{E}(f) \rightarrow \mathcal{E}(X)$  given by  $h[g_1, g_2] = [g_1]$  is an epimorphism with a section.

(2)  $f$  is a co- $\mathcal{E}$ -map if and only if  $h' : \mathcal{E}(f) \rightarrow \mathcal{E}(Y)$  given by  $h'[g_1, g_2] = [g_2]$  is an epimorphism with a section.

*Proof.* (1) Suppose that  $f$  is an  $\mathcal{E}$ -map. Then we have a map  $\phi_f : \mathcal{E}(X) \rightarrow \mathcal{E}(Y)$  such that  $\phi_f(g) \circ f \simeq f \circ g$  for any  $g \in \mathcal{E}(X)$ . Thus we have  $[g, \phi_f(g)] \in \mathcal{E}(f)$  and  $h[g, \phi_f(g)] = [g]$  and  $h$  is epimorphic. Next suppose that  $h$  is an epimorphism. For any  $[g] \in \mathcal{E}(X)$ , we have  $[g', g''] \in \mathcal{E}(f)$  such that  $h[g', g''] = [g]$ . So  $g$  is homotopic to  $g'$ . Since  $[g', g''] \in \mathcal{E}(f)$ ,  $g'$  and  $g''$  are homotopy equivalences and  $g'' \circ f \simeq f \circ g'$ . Thus we can define a map  $\phi_f : \mathcal{E}(X) \rightarrow \mathcal{E}(Y)$  by  $\phi_f(g) = \pi \circ s[g]$  where  $\pi : \mathcal{E}(f) \rightarrow \mathcal{E}(Y)$  is the natural projection and  $s$  is the section of the assumption. Hence,  $f$  is an  $\mathcal{E}$ -map.

(2) Suppose that  $f$  is a co- $\mathcal{E}$ -map. Then we have a map  $\psi_f : \mathcal{E}(Y) \rightarrow \mathcal{E}(X)$  such that  $g \circ f \simeq f \circ \psi_f(g)$  for any  $g \in \mathcal{E}(Y)$ . So we have  $[\psi_f(g), g] \in \mathcal{E}(f)$  and  $h'[\psi_f(g), g] = [g]$ . Thus  $h'$  is epimorphic. Next suppose that  $h'$  is an epimorphism. For any  $[g] \in \mathcal{E}(Y)$ , we have  $[g', g''] \in \mathcal{E}(f)$  such that  $h'[g', g''] = [g]$  and thus  $g$  is homotopic to  $g''$ . Since  $[g', g''] \in \mathcal{E}(f)$ ,  $g'$  and  $g''$  are homotopy equivalences and  $g'' \circ f \simeq f \circ g'$ . Then we can define a map  $\psi_f : \mathcal{E}(Y) \rightarrow \mathcal{E}(X)$  by  $\psi_f(g) = h \circ s'[g]$  for the section  $s'$ . Hence,  $f$  is a co- $\mathcal{E}$ -map.  $\square$

**Theorem 2.5.** Let  $\eta : S^3 \rightarrow S^2$  and  $\nu : S^7 \rightarrow S^4$  be the Hopf fibrations with fibre  $S^1$  and  $S^3$ , respectively. Let  $\epsilon_3 : S^{11} \rightarrow S^3$  be the generator of  $\pi_{11}(S^3) \cong \mathbb{Z}_2$  ([22]). Then

(1)  $\eta$  is a co- $\mathcal{E}$ -map, but not an  $\mathcal{E}$ -map,

- (2)  $\nu$  is neither an  $\mathcal{E}$ -map nor a co- $\mathcal{E}$ -map and  
 (3)  $\epsilon_3$  is both an  $\mathcal{E}$ -map and a co- $\mathcal{E}$ -map.

*Proof.* (1) From [17, Example 4.2 (i)], we have  $\Pi(\eta, \eta) = \{(k^2\iota_3, k\iota_2) \mid k \in \mathbb{Z}\}$  as a set. Therefore, we have a homotopy commutative diagram

$$\begin{array}{ccc} S^3 & \xrightarrow{k^2\iota_3} & S^3 \\ \eta \downarrow & & \downarrow \eta \\ S^2 & \xrightarrow{k\iota_2} & S^2 \end{array}$$

It is well known that  $\mathcal{E}(S^n) = \{\iota_n, -\iota_n\} \cong \mathbb{Z}_2$ . Since  $(\iota_3, -\iota_2), (\iota_3, \iota_2) \in \Pi(\eta, \eta)$ ,  $\eta$  is a co- $\mathcal{E}$ -map. However, there is no map  $f : S^2 \rightarrow S^2$  such that  $(-\iota_3, f) \in \Pi(\eta, \eta)$ . Thus  $\eta$  is not an  $\mathcal{E}$ -map.

(2) From [17, Example 4.2 (ii)], we have  $\Pi(\nu, \nu) = \{(k^2\iota_7, k\iota_4) \mid k(k-1) \equiv 0 \pmod{8}\}$  as a set. Therefore, we have a homotopy commutative diagram

$$\begin{array}{ccc} S^7 & \xrightarrow{k^2\iota_7} & S^7 \\ \nu \downarrow & & \downarrow \nu \\ S^4 & \xrightarrow{k\iota_4} & S^4 \end{array}$$

Since there are no maps  $f : S^7 \rightarrow S^7$  and  $g : S^4 \rightarrow S^4$  such that  $(f, -\iota_4), (-\iota_7, g) \in \Pi(\nu, \nu)$ ,  $\nu$  is neither an  $\mathcal{E}$ -map nor a co- $\mathcal{E}$ -map.

(3) From [17, Example 4.2 (iv)], we have  $\Pi(\epsilon_3, \epsilon_3) = \{(d+2s)\iota_{11}, d\iota_3 \mid d, s \in \mathbb{Z}\} \cong \mathbb{Z} \times \mathbb{Z}$  as a group. Therefore we have a homotopy commutative diagram

$$\begin{array}{ccc} S^{11} & \xrightarrow{(d+2s)\iota_{11}} & S^{11} \\ \epsilon_3 \downarrow & & \downarrow \epsilon_3 \\ S^3 & \xrightarrow{d\iota_3} & S^3 \end{array}$$

Since  $(\iota_{11}, \iota_3), (-\iota_{11}, -\iota_3) \in \Pi(\epsilon_3, \epsilon_3)$ ,  $\epsilon_3$  is both an  $\mathcal{E}$ -map and a co- $\mathcal{E}$ -map.  $\square$

**Example 2.6.** (1) Let  $e : X \rightarrow \Omega\Sigma X$  be the adjoint of  $id_{\Sigma X}$  from the one-to-one correspondence  $[X, \Omega\Sigma X] \cong [\Sigma X, \Sigma X]$ . We know that  $e(x)(t) = \langle x, t \rangle$ . Let  $f$  be a self homotopy equivalence on  $X$ , that is,  $f \in \mathcal{E}(X)$  and let  $f'$  be a homotopy inverse of  $f$ . It is clear that the map  $\Sigma f : \Sigma X \rightarrow \Sigma X$ ,  $\Sigma f \langle x, t \rangle = \langle f(x), t \rangle$ , is a homotopy equivalence with homotopy inverse  $\Sigma f'$ . Then we define a map  $\tilde{f} : \Omega\Sigma X \rightarrow \Omega\Sigma X$  by  $\tilde{f}(\alpha)(t) = \Sigma f(\alpha(t))$ . Define another map  $\tilde{f}' : \Omega\Sigma X \rightarrow \Omega\Sigma X$  by  $\tilde{f}'(\alpha)(t) = \Sigma f'(\alpha(t))$ . Clearly we have  $\tilde{f} \circ \tilde{f}' \simeq id$  and  $\tilde{f}' \circ \tilde{f} \simeq id$ . Moreover we have  $e(f(x))(t) = \langle f(x), t \rangle$  and  $\tilde{f}(e(x))(t) = \Sigma f(e(x)(t)) = \Sigma f \langle x, t \rangle = \langle f(x), t \rangle$ . Therefore we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ e \downarrow & & \downarrow e \\ \Omega\Sigma X & \xrightarrow{\tilde{f}} & \Omega\Sigma X \end{array}$$

Thus  $e : X \rightarrow \Omega\Sigma X$  is an  $\mathcal{E}$ -map.

(2) Let  $\pi : \Sigma\Omega Y \rightarrow Y$  be the adjoint of  $id_{\Omega Y}$  from the one-to-one correspondence  $[\Sigma\Omega Y, Y] \cong [\Omega Y, \Omega Y]$ . We know that  $\pi\langle\alpha, t\rangle = \alpha(t)$ . Let  $g$  be a self homotopy equivalence on  $Y$ , that is  $g \in \mathcal{E}(Y)$  and let  $g'$  be a homotopy inverse of  $g$ . Then we define a map  $\tilde{g} : \Sigma\Omega Y \rightarrow \Sigma\Omega Y$  by  $\tilde{g}\langle\alpha, t\rangle = \langle g \circ \alpha, t\rangle$  and  $\tilde{g}' : \Sigma\Omega Y \rightarrow \Sigma\Omega Y$  by  $\tilde{g}'\langle\alpha, t\rangle = \langle g' \circ \alpha, t\rangle$ . Clearly we have  $\tilde{g} \circ \tilde{g}' \simeq id$  and  $\tilde{g}' \circ \tilde{g} \simeq id$ . Moreover we have  $(\pi \circ \tilde{g})\langle\alpha, t\rangle = \pi\langle g \circ \alpha, t\rangle = (g \circ \alpha)(t)$  and  $(g \circ \pi)\langle\alpha, t\rangle = g(\alpha(t))$ . Therefore we have a commutative diagram

$$\begin{array}{ccc} \Sigma\Omega Y & \xrightarrow{\tilde{g}} & \Sigma\Omega Y \\ \pi \downarrow & & \downarrow \pi \\ Y & \xrightarrow{g} & Y \end{array}$$

Therefore  $\pi : \Sigma\Omega Y \rightarrow Y$  is a co- $\mathcal{E}$ -map.

**Example 2.7.** There is a natural homomorphism  $\mathcal{E}(X_{(n)}) \rightarrow \mathcal{E}(X_{(n-1)})$  obtained by restricting the map to a lower Postnikov section [3, p.27]. Thus the principal  $K(\pi_n(X), n)$ -fibration  $X_{(n)} \rightarrow X_{(n-1)}$  is an  $\mathcal{E}$ -map. The restriction map  $X \rightarrow X_{(n-1)}$  is also an  $\mathcal{E}$ -map. On the other hand, for the  $n$ -skeleton  $X^{(n)}$ , the inclusions  $X^{(n)} \rightarrow X^{(n+1)}$  and  $X^{(n)} \rightarrow X$  are both co- $\mathcal{E}$ -maps.

**Remark 2.8.** Recall that a space  $X$  is said to be (homotopically) *rigid* when  $\mathcal{E}(X) = \{id_X\}$  [9] ([4]). If  $X$  is rigid, then every map  $f : X \rightarrow Y$  is an  $\mathcal{E}$ -map by  $\phi_f(id_X) = id_Y$ . If  $Y$  is rigid, then every map  $f : X \rightarrow Y$  is a co- $\mathcal{E}$ -map by  $\psi_f(id_Y) = id_X$ . Also we can construct infinitely many examples of  $\mathcal{E}$ -maps and co- $\mathcal{E}$ -maps by using the functor of [9, Remark 2.8].

### 3. COMPUTATIONS IN SULLIVAN MODELS

We assume that  $X$  is a nilpotent CW complex. Let  $M(X) = (\Lambda V, d)$  be the Sullivan minimal model of  $X$  [21]. It is a free  $\mathbb{Q}$ -commutative differential graded algebra (DGA) with a  $\mathbb{Q}$ -graded vector space  $V = \bigoplus_{i \geq 1} V^i$  where  $\dim V^i < \infty$  and a decomposable differential; i.e.,  $d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1}$  and  $d \circ d = 0$ . Here  $\Lambda^+ V$  is the ideal of  $\Lambda V$  generated by elements of positive degree. The degree of a homogeneous element  $x$  of a graded algebra is denoted as  $|x|$ . Then  $xy = (-1)^{|x||y|}yx$  and  $d(xy) = d(x)y + (-1)^{|x|}xd(y)$ . Note that  $M(X)$  determines the rational homotopy type of  $X$ . In particular,  $H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q})$  and  $V^i \cong Hom(\pi_i(X), \mathbb{Q})$ . Refer to [10] for details.

Let  $\text{Aut} M$  be the group of DGA-automorphisms of a DGA  $M$ . For a nilpotent space  $X$  and a (not necessarily minimal) model  $M(X)$ , there is a group isomorphism

$$\mathcal{E}(X_{\mathbb{Q}}) \cong \mathcal{E}(M(X)),$$

where  $\mathcal{E}(M(X)) = \text{Aut} M(X) / \sim$  is the group of self-DGA-homotopy equivalence classes of  $M(X)$  [21]. Now recall about “DGA-homotopy” in [12]: In general, two maps  $f : M(Y) \rightarrow M(X)$  and  $g : M(Y) \rightarrow M(X)$  are DGA-homotopic (denote as  $f \sim g$ ) if there is a DGA-map  $H : M(Y) \rightarrow M(X) \otimes \Lambda(t, dt)$  such that  $H|_{t=0, dt=0} = f$  and  $H|_{t=1, dt=0} = g$ . Here  $|t| = 0$  and  $|dt| = 1$  with  $d(t) = dt$ ,  $d(dt) = 0$ .

The group  $\mathcal{E}(M(X))$  does not depend on choosing a model of  $X$ . For example, the minimal model  $M = M(S^{2n+1}) = (\Lambda w, 0)$  and a non-minimal model  $M' = (\Lambda(y, w, v), D)$  with  $|y| = 2$ ,  $|w| = 2n+1$ ,  $|v| = 1$ ,  $Dy = 0$ ,  $Dw = y^{n+1}$  and  $Dv = y$  are both models of  $S^{2n+1}$ . Obviously we have  $\mathcal{E}(M) \cong \mathbb{Q}^*$  by  $w \rightarrow aw$  for

$a \in \mathbb{Q}^*$ . On the other hand, in  $\mathcal{E}(M')$  we can set  $H(y) = cyt + cvdt$ ,  $H(v) = cvt$  and  $H(w) = aw + by^n v + b'y^n vt^{n+1}$  with  $a, b, b', c \in \mathbb{Q}^*$ . Then  $a + b = 0$  and  $b' = c^{n+1}$  from  $D \circ H = H \circ D$ . Thus two maps  $f \in \text{Aut } M'$  given by  $f(y) = f(v) = 0$ ,  $f(w) = a(w - y^n v)$  and  $g \in \text{Aut } M'$  given by  $g(y) = cy$ ,  $g(v) = cv$ ,  $g(w) = a(w - y^n v) + c^{n+1}y^n v$  are DGA-homotopic. Hence we have  $\mathcal{E}(M') = \mathcal{E}(M) \cong \mathbb{Q}^*$  as in Example 3.3(1) below.

**Remark 3.1.** From the universality of the localization [15], the rationalization map  $l : X \rightarrow X_{\mathbb{Q}}$  is an  $\mathcal{E}$ -map, but it is not a co- $\mathcal{E}$ -map in general. For example, when  $X = S^3$ , the elements  $f$  of  $\mathcal{E}(M(X)) = \mathcal{E}(\Lambda(x), 0)$  with  $f(x) = ax$  for  $a \neq \pm 1 \in \mathbb{Q}^* = \mathbb{Q} - 0$  can not be realized as a homotopy equivalence of  $X$ .

The model of a map  $f : X \rightarrow Y$  between nilpotent spaces is given by a relative model:

$$M(Y) = (\Lambda W, d_Y) \xrightarrow{i} (\Lambda W \otimes \Lambda V, D) \xrightarrow{q} (\Lambda V, \overline{D})$$

with  $D|_{\Lambda W} = d_Y$  and the minimal model  $(\Lambda V, \overline{D})$  of the homotopy fiber of  $f$ . It is well known that there is a quasi-isomorphism  $M(X) \rightarrow (\Lambda W \otimes \Lambda V, D)$  [10]. Then Definition 1.2 is translated to

**Definition 3.2.** Let  $f : X \rightarrow Y$  be a map between nilpotent spaces.

(1) The map  $f$  is a rational  $\mathcal{E}$ -map if and only if there is a homomorphism  $\phi_f : \mathcal{E}(\Lambda W \otimes \Lambda V, D) \rightarrow \mathcal{E}(\Lambda W, d_Y)$  such that

$$\begin{array}{ccc} (\Lambda W \otimes \Lambda V, D) & \xrightarrow{g} & (\Lambda W \otimes \Lambda V, D) \\ \uparrow i & & \uparrow i \\ (\Lambda W, d_Y) & \xrightarrow{\phi_f(g)} & (\Lambda W, d_Y) \end{array}$$

is DGA-homotopy commutative for any element  $g$  of  $\mathcal{E}(\Lambda W \otimes \Lambda V, D)$ .

(2) The map  $f$  is a rational co- $\mathcal{E}$ -map if and only if there is a homomorphism  $\psi_f : \mathcal{E}(\Lambda W, d_Y) \rightarrow \mathcal{E}(\Lambda W \otimes \Lambda V, D)$  such that

$$\begin{array}{ccc} (\Lambda W \otimes \Lambda V, D) & \xrightarrow{\psi_f(g)} & (\Lambda W \otimes \Lambda V, D) \\ \uparrow i & & \uparrow i \\ (\Lambda W, d_Y) & \xrightarrow{g} & (\Lambda W, d_Y) \end{array}$$

is DGA-homotopy commutative for any element  $g$  of  $\mathcal{E}(\Lambda W, d_Y)$ .

**Example 3.3.** (1) For the Hopf fibration  $S^1 \rightarrow S^{2n+1} \xrightarrow{f} \mathbb{C}P^n$ , the relative model is given by

$$(\Lambda(y, w), d_Y) \rightarrow (\Lambda(y, w, v), D) \rightarrow (\Lambda(v), 0)$$

with  $|y| = 2$ ,  $|w| = 2n + 1$ ,  $|v| = 1$ ,  $d_Y w = y^{n+1}$  and  $Dv = y$ . We can identify  $\mathcal{E}(\mathbb{C}P^n_{\mathbb{Q}})$  as  $\mathbb{Q}^*$  by  $g(y) = ay$  and  $g(w) = a^{n+1}w$  for  $g \in \mathcal{E}(\mathbb{C}P^n_{\mathbb{Q}})$  and  $a \in \mathbb{Q}^*$ . Also we have  $\mathcal{E}(S^{2n+1}_{\mathbb{Q}}) = \mathcal{E}(\Lambda(y, w, v), D) = \mathcal{E}(\Lambda w, 0) \cong \mathbb{Q}^*$ . Then there is a homomorphism

$$\psi_f : \mathbb{Q}^* \cong \mathcal{E}(\mathbb{C}P^n_{\mathbb{Q}}) \rightarrow \mathcal{E}(S^{2n+1}_{\mathbb{Q}}) \cong \mathbb{Q}^*$$

which is given by  $\psi_f(a) = a^{n+1}$  for  $a \in \mathbb{Q}^*$ . Thus  $f$  is a rational co- $\mathcal{E}$ -map, but it is not a rational  $\mathcal{E}$ -map.

(2) Let  $X$  be the pullback of the sphere bundle of the tangent bundle of  $S^{m+n}$  by the canonical degree 1 map  $S^m \times S^n \rightarrow S^{m+n}$  for odd integers  $m$  and  $n$ . Then it is the total space of a fibration  $S^{m+n-1} \rightarrow X \xrightarrow{f} S^m \times S^n$  whose model is

$$(\Lambda(w_1, w_2), 0) \rightarrow (\Lambda(w_1, w_2, u), D) \rightarrow (\Lambda(u), 0)$$

with  $|w_1| = m$ ,  $|w_2| = n$ ,  $|u| = m + n - 1$  and  $Du = w_1 w_2$  is both a rational  $\mathcal{E}$ -map and a rational co- $\mathcal{E}$ -map.

(3) The fibration  $S^m \times S^{m+n-1} \rightarrow X \xrightarrow{f} S^n$  ( $m \neq n$  are odd) whose model is

$$(\Lambda(w), 0) \rightarrow (\Lambda(w, v, u), D) \rightarrow (\Lambda(v, u), 0)$$

where  $|w| = n$ ,  $|v| = m$ ,  $|u| = m + n - 1$  and  $Du = wv$  with  $m, n$  odd is both a rational  $\mathcal{E}$ -map and a rational co- $\mathcal{E}$ -map.

(4) For the fibration  $\mathbb{C}P^{n-1} \rightarrow \mathbb{C}P^{2n-1} \xrightarrow{f} S^{2n}$  given by

$$(\Lambda(y, w), d_Y) \rightarrow (\Lambda(y, w, x, v), D) \rightarrow (\Lambda(x, v), \overline{D})$$

with  $|y| = 2n$ ,  $|w| = 4n - 1$ ,  $|x| = 2$ ,  $|v| = 2n - 1$ ,  $d_Y y = 0$ ,  $d_Y w = y^2$ ,  $Dx = \overline{D}x = 0$ ,  $Dv = y - x^n$  and  $\overline{D}v = x^n$ , the map  $f$  is a rational  $\mathcal{E}$ -map given by  $\phi_f(a) = a^n$  for  $a \in \mathbb{Q}^*$  but not a rational co- $\mathcal{E}$ -map.

**Example 3.4.** For an  $n$ -dimensional manifold  $X$ , the collapsing map of lower cells  $f : X \rightarrow S^n$  is an  $\mathcal{E}$ -map. Indeed, from the commutative diagram between cofibrations

$$\begin{array}{ccc} X^{(n-1)} & \xrightarrow{g|_{X^{(n-1)}}} & X^{(n-1)} \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & X \\ \downarrow f & & \downarrow f \\ S^n & \xrightarrow{\quad \overline{g} \quad} & S^n, \end{array}$$

we have  $\phi_f(g) = \overline{g}$ , but it is not a (rational) co- $\mathcal{E}$ -map in general. For example, the collapsing map of lower cells  $f : X = \mathbb{C}P^n \rightarrow S^{2n} = Y$  induces a DGA-map

$$f^* : M(Y) = (\Lambda(y, w), d_Y) \rightarrow (\Lambda(x, v), d_X) = M(X)$$

with  $d_Y w = y^2$ ,  $d_X v = x^{n+1}$ ,  $f^*(y) = x^n$  and  $f^*(w) = x^{n-1}v$ . The map  $f$  is a rational  $\mathcal{E}$ -map by  $\phi_f(a) = a^n$  for  $a \in \mathbb{Q}^*$  but not a rational co- $\mathcal{E}$ -map. Indeed, for  $g^*(y) = ay$  with  $a \notin (\mathbb{Q}^*)^{\times m} := \mathbb{Q}^* \cdot \mathbb{Q}^* \cdots \mathbb{Q}^*$  ( $m$ -times), we cannot define  $\psi_{f^*}(g^*)$ .

**Example 3.5.** Let  $\Omega Y = \text{map}((S^1, *), (Y, *))$  be the base point preserving the loop space of a simply connected space  $Y$  and  $LY = \text{map}(S^1, Y)$ , the free loop space of  $Y$ . We consider the evaluation map  $f : LY \rightarrow Y$  with  $f(\sigma) = \sigma(*)$ . It is a co- $\mathcal{E}$ -map by  $\psi_f(g)(h) = g \circ h$  for  $g \in \mathcal{E}(Y)$ . What is the (rational) homotopical condition of  $Y$  that allows  $f$  to be a (rational)  $\mathcal{E}$ -map? According to [23], the relative model of the free loop fibration  $\Omega Y \rightarrow LY \xrightarrow{f} Y$ :

$$M(Y) = (\Lambda V, d) \rightarrow (\Lambda V \otimes \Lambda \overline{V}, D) \rightarrow (\Lambda \overline{V}, 0)$$

with  $M(LY) = (\Lambda V \otimes \Lambda \overline{V}, D)$  is defined as follow: The graded vector space  $\overline{V}$  satisfies  $\overline{V}^n \cong V^{n-1}$  for  $n > 0$  and denote by  $s : V \rightarrow \overline{V}$  ( $s(v) := \overline{v}$ ) this isomorphism of degree  $-1$ . There is a unique extension of  $s$  into a derivation of algebra



$s : \Lambda V \otimes \Lambda \overline{V} \rightarrow \Lambda V \otimes \Lambda \overline{V}$  such that  $s(\overline{V}) = 0$ . The differential  $D$  is given by  $D(v) = d(v)$  for  $v \in V$  and  $D(\overline{v}) = -s \circ d(v)$  for  $\overline{v} \in \overline{V}$ .

If every DGA-isomorphism  $g$  of  $(\Lambda V \otimes \Lambda \overline{V}, D)$  satisfies  $g|_{\Lambda V} \in \mathcal{E}(\Lambda V, d)$ , then  $f(M(f))$  is a rational  $\mathcal{E}$ -map by  $\phi_f(g) = g|_{\Lambda V}$ .

(1) When  $Y = S^n$ , we observe that the map  $f$  is a rational  $\mathcal{E}$ -map. If  $n$  is even,  $M(S^n) = (\Lambda(x, y), d)$  with  $|x| = n$ ,  $|y| = 2n + 1$ ,  $dx = 0$  and  $dy = x^2$ . For example, when  $n = 2$ , note that there is no DGA-map  $g(x) = x + \overline{y}$ .

(2) When  $Y = S^m \times S^n$  for odd integers  $m < n$ , the map  $f$  is a rational  $\mathcal{E}$ -map if and only if  $m - 1$  is not a divisor of  $n - 1$ . Indeed, let  $M(S^m \times S^n) = (\Lambda(x, y), 0)$ . When  $n - 1 = a(m - 1)$  for an integer  $a > 1$ , there is a DGA-isomorphism  $g : (\Lambda(x, y, \overline{x}, \overline{y}), 0) \rightarrow (\Lambda(x, y, \overline{x}, \overline{y}), 0)$  with  $g(x) = x$ ,  $g(\overline{x}) = \overline{x}$ ,  $g(\overline{y}) = \overline{y}$  and  $g(y) = y + \overline{x}^{a-1}x$ . Then  $f$  cannot be a rational  $\mathcal{E}$ -map. When  $n - 1 \neq a(m - 1)$  for any  $a$ , a self-map  $g$  is given by  $g(x) = x$  and  $g(y) = y$  from the degree reason.

*Proof of Theorem 1.4(1).* Note that  $\pi_*(j)_{\mathbb{Q}}$  is injective if and only if the model of  $j : H \rightarrow G$  is given as the projection  $M(G) \cong (\Lambda(v_1, \dots, v_k, u_1, \dots, u_l), 0) \rightarrow (\Lambda(v_1, \dots, v_k), 0) \cong M(H)$  after a suitable basis change. Then we can define as  $\phi_j(g) = g \otimes 1_{\Lambda(u_1, \dots, u_l)}$  for any  $g \in \mathcal{E}(\Lambda(v_1, \dots, v_k), 0)$ .  $\square$

For the  $n$ -dimensional unitary group  $U(n)$ ,  $M(U(n)) = M(S^1 \times \dots \times S^{2n-1}) = (\Lambda(v_1, \dots, v_n), 0)$  with  $|v_i| = 2i - 1$ . For the  $n$ -dimensional special unitary group  $SU(n)$ ,  $M(SU(n)) = (\Lambda(v_1, \dots, v_{n-1}), 0)$  with  $|v_i| = 2i + 1$ . For the  $n$ -dimensional symplectic group  $Sp(n)$ ,  $M(Sp(n)) = (\Lambda(v_1, \dots, v_n), 0)$  with  $|v_i| = 4i - 1$ .

**Example 3.6.** In general, for a connected closed sub-Lie group  $H$  of a compact connected Lie group  $G$ , the inclusion  $j : H \rightarrow G$ , is not a rational  $\mathcal{E}$ -map. For example, the blockwise inclusion  $j : SU(3) \times SU(3) \rightarrow SU(6)$  is not. Indeed,  $M(SU(3) \times SU(3)) = (\Lambda(u_1, w_1, u_2, w_2), 0)$  with  $|u_1| = |w_1| = 3$ ,  $|u_2| = |w_2| = 5$  and  $M(SU(6)) = (\Lambda(v_1, v_2, v_3, v_4, v_5), 0)$  with  $|v_i| = 2i + 1$ .  $M(j)(v_i) = u_i + w_i$  for  $i = 1, 2$ . Then we cannot define  $\phi_j(g)$  for  $g \in \mathcal{E}(\Lambda(u_1, w_1, u_2, w_2), 0)$  when  $g(u_i) = u_i$ ,  $g(w_i) = -w_i$  for example.

**Lemma 3.7.** Let  $X = S^{a_1} \times \dots \times S^{a_m} \times Y$  and  $Y = S^{b_1} \times \dots \times S^{b_n}$  for odd-integers  $a_1 \leq \dots \leq a_m \leq b_1 \leq \dots \leq b_n$ . Then the second factor projection map  $f : X \rightarrow Y$  is a rational  $\mathcal{E}$ -map if and only if there are no subsets  $\{i_1, \dots, i_k\}$  of  $\{1, \dots, m\}$  and  $\{j_1, \dots, j_k\}$  of  $\{1, \dots, n\}$  with  $b_k = a_{i_1} + \dots + a_{i_k} + b_{j_1} + \dots + b_{j_k}$  for  $k = 1, \dots, n$ .

*Proof.* Put  $M(X) = (\Lambda(x_1, \dots, x_m, y_1, \dots, y_n), 0)$  and  $M(Y) = (\Lambda(y_1, \dots, y_n), 0)$  with  $|x_i| = a_i$  and  $|y_i| = b_i$ . If  $b_k = a_{i_1} + \dots + a_{i_k} + b_{j_1} + \dots + b_{j_k}$ , there is a map  $g \in \mathcal{E}(M(X))$  such that

$$g(x_i) = x_i \quad (i \leq m), \quad g(y_i) = y_i \quad (i \neq k), \quad g(y_k) = y_k + x_{i_1} \cdots x_{i_k} y_{j_1} \cdots y_{j_k}$$

and  $M(f)(y_i) = y_i$  for all  $i$ . Then we can not have a DGA-homotopy commutative diagram

$$\begin{array}{ccc} (\Lambda(x_1, \dots, x_m, y_1, \dots, y_n), 0) & \xrightarrow{g} & (\Lambda(x_1, \dots, x_m, y_1, \dots, y_n), 0) \\ \uparrow M(f) & & \uparrow M(f) \\ (\Lambda(y_1, \dots, y_n), 0) & \xrightarrow{\phi_f(g)} & (\Lambda(y_1, \dots, y_n), 0). \end{array}$$

If  $b_k \neq a_{i_1} + \cdots + b_{j_k}$  for any  $k$  and index set, we can put

$$\phi_f(g) = g \mid_{\Lambda(y_1, \dots, y_n)}$$

in the diagram for any map  $g \in \mathcal{E}(M(X))$ .  $\square$

**Theorem 3.8.** (1) When  $2 < m < n$ , the natural projection  $p_{n,m} : U(n) \rightarrow U(n)/U(m)$  is a rational  $\mathcal{E}$ -map if and only if  $n < 5$ .

(2) When  $2 < m < n$ , the natural projection  $p_{n,m} : SU(n) \rightarrow SU(n)/SU(m)$  is a rational  $\mathcal{E}$ -map if and only if  $n < 8$ .

**Lemma 3.9.** Let  $X = S^{a_1} \times \cdots \times S^{a_m}$  and  $Y = X \times S^{b_1} \times \cdots \times S^{b_n}$  for odd-integers  $a_1 \leq \cdots \leq a_m \leq b_1 \leq \cdots \leq b_n$ . Then the first factor inclusion map  $f : X \rightarrow Y$  is a rational co- $\mathcal{E}$ -map if and only if there is no subset  $\{i_1, \dots, i_k\}$  of  $\{1, \dots, m\}$  with  $b_k = a_{i_1} + \cdots + a_{i_k}$  for  $k = 1, \dots, n$ .

*Proof.* Let  $M(X) = (\Lambda(x_1, \dots, x_m), 0)$  and  $M(Y) = (\Lambda(x_1, \dots, x_m, y_1, \dots, y_n), 0)$  with  $|x_i| = a_i$  and  $|y_i| = b_i$ . If  $b_k = a_{i_1} + \cdots + a_{i_k}$ , there is a map  $g \in \mathcal{E}(M(Y))$  such that

$$g(x_i) = x_i \quad (i \leq m), \quad g(y_i) = y_i \quad (i \neq k), \quad g(y_k) = y_k + x_{i_1} \cdots x_{i_k}$$

and  $M(f)(x_i) = x_i$  and  $M(f)(y_i) = 0$  for all  $i$ . Then we cannot have a DGA-homotopy commutative diagram

$$\begin{array}{ccc} (\Lambda(x_1, \dots, x_m), 0) & \xrightarrow{\psi_f(g)} & (\Lambda(x_1, \dots, x_m), 0) \\ M(f) \uparrow & & \uparrow M(f) \\ (\Lambda(x_1, \dots, x_m, y_1, \dots, y_n), 0) & \xrightarrow{g} & (\Lambda(x_1, \dots, x_m, y_1, \dots, y_n), 0). \end{array}$$

If  $b_k \neq a_{i_1} + \cdots + a_{i_k}$  for any  $k$  and  $\{i_1, \dots, i_k\}$ , we can put

$$\psi_f(g) = g \mid_{\Lambda(x_1, \dots, x_m)}$$

in the diagram for any map  $g \in \mathcal{E}(M(Y))$ .  $\square$

From Lemma 3.9, we have the following.

**Theorem 3.10.** (1) When  $2 < m < n$ , the natural inclusion map  $i_{m,n} : U(m) \rightarrow U(n)$  is a rational co- $\mathcal{E}$ -map if and only if  $n < 5$ .

(2) When  $2 < m < n$ , the natural inclusion map  $i_{m,n} : SU(m) \rightarrow SU(n)$  is a rational co- $\mathcal{E}$ -map if and only if  $n < 8$ .

(3) When  $m \leq 4$ , the natural inclusion map  $i_{m,n} : Sp(m) \rightarrow Sp(n)$  is a rational co- $\mathcal{E}$ -map for any  $m \leq n$ . When  $4 < m < n$ , the natural inclusion map  $i_{m,n} : Sp(m) \rightarrow Sp(n)$  is a rational co- $\mathcal{E}$ -map if and only if  $n < 14$ .

*Proof.* (3) For  $S = \{3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 55, \dots\}$ , there are no integers  $a, b, c, d \in S$  with  $a < b < c < d$  satisfying the equation  $a + b + c = d$  since

$$(4i - 1) + (4j - 1) + (4k - 1) = 4(i + j + k) - 3 \neq 4l - 1$$

for any  $i, j, k, l \in \mathbb{N}$ . On the other hand,  $3 + 7 + 11 + 15 + (19 + 4i) = 55 + 4i = |v_{14+i}|$  for  $i \geq 0$ .  $\square$

For a connected closed sub-Lie group  $H$  of a compact connected Lie group  $G$  with inclusion  $j : H \rightarrow G$ , there is the induced map  $Bj : BH \rightarrow BG$  between the classifying spaces. It induces a map  $Bj^* : M(BG) = (\Lambda V_{BG}, 0) = (\mathbb{Q}[x_1, \dots, x_k], 0) \rightarrow$

$(\Lambda V_{BH}, 0) = M(BH)$  between the models. Here  $|x_i|$  are even and  $\text{rank} G = k$ . Let  $V_G^n = V_{BG}^{n+1}$  by corresponding  $y_i$  to  $x_i$  with  $|y_i| = |x_i| - 1$ .

**Lemma 3.11.** ([10, Proposition 15.16]) *The (non-minimal) model of  $G/H$  is given as  $(\Lambda V_{BH} \otimes \Lambda V_G, d)$  where  $dx_i = 0$  and  $dy_i = Bj^*(x_i)$  for  $i = 1, \dots, k$ .*

*Proof of Theorem 1.4(2).* For  $f : G \rightarrow G/H$ ,  $M(f)$  is given by the projection  $(\Lambda V_{BH} \otimes \Lambda V_G, d) \rightarrow (\Lambda V_G, 0)$  sending elements of  $\Lambda V_{BH}$  to zero from Lemma 3.11. Thus we can define  $\psi_f(g)$  for any  $g \in \mathcal{E}(\Lambda V_{BH} \otimes \Lambda V_G, d)$  by  $\psi_f(g) = \bar{g}$  because  $g(x_i) \in \mathbb{Q}[x_1, \dots, x_k]$ .  $\square$

**Example 3.12.** Let  $X$  be a  $G$ -space for a Lie group  $G$ . When is the orbit map  $f : X \rightarrow X/G$  a rational co- $\mathcal{E}$ -map? Let  $X = S^2 \times S^3$ , where  $M(S^2 \times S^3) = (\Lambda(x, y, z), d)$  with  $dx = dz = 0$  and  $dy = x^2$  of  $|x| = 2$ ,  $|y| = |z| = 3$ . There are free  $S^1$ -actions on  $X$  where  $M(X/S^1) = M(ES^1 \times_{S^1} X) = (\Lambda(t, x, y, z), D)$  for  $M(BS^1) = (\mathbb{Q}[t], 0)$  with  $|t| = 2$  [1], [13]. If the Borel space of a  $S^1$ -action has the model with  $Dx = Dt = 0$ ,  $Dy = x^2$  and  $Dz = t^2$  (it is given by a free action on  $S^3$ ),  $f$  is not a rational co- $\mathcal{E}$ -map. Indeed, we can not define  $\psi_f(g)$  for the DGA-map  $g$  with  $g(x) = t$ ,  $g(t) = x$ ,  $g(y) = z$  and  $g(z) = y$ . But if a  $S^1$ -action has the model with  $Dy = x^2 + at^2$  and  $Dz = xt$  for  $a \notin \mathbb{Q}^*/(\mathbb{Q}^*)^2$ , the orbit map  $f$  is a rational co- $\mathcal{E}$ -map.

**Remark 3.13.** Even if a map  $f$  is an  $\mathcal{E}$ -map, it may not be a rational  $\mathcal{E}$ -map. Recall an example of Arkowitz and Lupton [4]: Let  $Y$  be a rational (non-universal) space such that  $M(Y) = (\Lambda(x_1, x_2, y_1, y_2, y_3, z), d)$  with  $|x_1| = 10$ ,  $|x_2| = 12$ ,  $|y_1| = 41$ ,  $|y_2| = 43$ ,  $|y_3| = 45$ ,  $|z| = 119$ ,

$$dx_1 = dx_2 = 0, \quad dy_1 = x_1^3 x_2, \quad dy_2 = x_1^2 x_2^2, \quad dy_3 = x_1 x_2^3 \quad \text{and}$$

$$dz = x_2(y_1 x_2 - x_1 y_2)(y_2 x_2 - x_1 y_3) + x_1^{12} + x_2^{10}.$$

Then  $\mathcal{E}(Y) = \{g_1, g_2\} (\cong \{1, -1\})$  where  $g_1 = id_Y$  and  $g_2$  is given by

$$g_2(x_1) = x_1, \quad g_2(x_2) = -x_2, \quad g_2(y_1) = -y_1,$$

$$g_2(y_2) = y_2, \quad g_2(y_3) = -y_3, \quad g_2(z) = z$$

[4, Example 5.2]. Consider the 12-dimensional homotopy generator  $f : S^{12} \rightarrow Y$  corresponding to  $x_2$ . It is an  $\mathcal{E}$ -map by the homotopy commutative diagram:

$$\begin{array}{ccc} S^{12} & \xrightarrow{\pm 1} & S^{12} \\ f \downarrow & & \downarrow f \\ Y & \xrightarrow{\phi_f(\pm 1)} & Y \end{array}$$

where  $\phi_f : \mathcal{E}(S^{12}) = \{\pm 1\} \cong \mathcal{E}(Y)$  by  $\phi_f(1) = g_1$  and  $\phi_f(-1) = g_2$ . But it is not a rational  $\mathcal{E}$ -map. Because there is no map  $\phi_f : \mathcal{E}(M(S^{12})) = \mathcal{E}(\Lambda(u, v), d) \rightarrow \mathcal{E}(M(Y))$  when  $a \neq \pm 1 \in \mathbb{Q}^*$ , i.e., there is no homotopy commutative diagram:

$$\begin{array}{ccc} (\Lambda(u, v), d) & \xrightarrow{\times a} & (\Lambda(u, v), d) \\ M(f) \uparrow & & \uparrow M(f) \\ M(Y) & \xrightarrow{\phi_f(\times a)} & M(Y) \end{array}$$

where  $M(f)(x_2) = u$ ,  $M(f)(z) = u^8 v$  and  $M(f)$  sends the others to zero. Here  $M(S^{12}) = (\Lambda(u, v), d)$  with  $|u| = 12$ ,  $|v| = 23$ ,  $du = 0$ ,  $dv = u^2$  and  $\times a(u) = au$ ,  $\times a(v) = a^2 v$ .

*Proof of Theorem 1.7* (1) It is obvious from the definition.

(2) As (non-graded) DGAs,  $M(\mathbb{C}P^n) \cong M(\mathbb{H}P^n) \cong (\Lambda(x, y), d)$  where  $dx = 0$  and  $dy = x^{n+1}$ . Therefore the inclusions  $S^2 \rightarrow \mathbb{C}P^n$  and  $S^4 \rightarrow \mathbb{H}P^n$  induce  $\mathcal{E}(S_{\mathbb{Q}}^2) \cong \mathcal{E}(\mathbb{C}P^n_{\mathbb{Q}}) (\cong \mathbb{Q}^* := \mathbb{Q} - 0)$  and  $\mathcal{E}(S_{\mathbb{Q}}^4) \cong \mathcal{E}(\mathbb{H}P^n_{\mathbb{Q}}) (\cong \mathbb{Q}^*)$ , respectively.

(3) Suppose that  $m$  is even and  $n$  is odd. Then  $M(S^m \times S^n) = (\Lambda(x, y, z), d)$  where  $dx = dy = 0$  and  $dz = x^2$  with  $|x| = m$ ,  $|y| = n$  and  $|z| = 2m - 1$ .

If  $m - 1 = n$ , we have  $|z| = |xy|$ . Then any element of  $\mathcal{E}(\Lambda(x, y, z), d)$  is given as

$$x \rightarrow ax, \quad y \rightarrow by, \quad z \rightarrow a^2 z + cxy$$

for some  $a, b \in \mathbb{Q}^*$  and  $c \in \mathbb{Q}$ . The same is not true for  $\mathcal{E}((S^m \vee S^n)_{\mathbb{Q}}) = \mathcal{E}((\Lambda(x, y, z, \dots), d)$  since  $[xy] = 0$  in  $H^*(S^m \vee S^n; \mathbb{Q}) = \mathbb{Q}[x] \otimes \Lambda(y)/(x^2, xy)$ , i.e.,  $c = 0$ .

If  $2m - 1 = n$ , we have  $|z| = |y|$ . Then any element of  $\mathcal{E}(\Lambda(x, y, z), d)$  is given as

$$x \rightarrow ax, \quad y \rightarrow by, \quad z \rightarrow a^2 z + cy$$

for some  $a, b \in \mathbb{Q}^*$  and  $c \in \mathbb{Q}$ . The same is true for  $\mathcal{E}((S^m \vee S^n)_{\mathbb{Q}}) = \mathcal{E}((\Lambda(x, y, z, \dots), d)$ , which is also checked by using its Quillen model [10], [8].

In the other case,  $c = 0$  for  $S^m \vee S^n$  and  $S^m \times S^n$ .

Thus we have the following table:

$\mathcal{E}((S^m \vee S^n)_{\mathbb{Q}})$	$\mathcal{E}((S^m \times S^n)_{\mathbb{Q}})$	$m:\text{even}, n:\text{odd}$
$\mathbb{Q}^* \times \mathbb{Q}^*$	$\mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}$	$m - 1 = n$
$\mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}$	$\mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}$	$2m - 1 = n$
$\mathbb{Q}^* \times \mathbb{Q}^*$	$\mathbb{Q}^* \times \mathbb{Q}^*$	other

Hence the inclusion  $f : S^m \vee S^n \rightarrow S^m \times S^n$  induces  $\phi_f : \mathcal{E}((S^m \vee S^n)_{\mathbb{Q}}) \cong \mathcal{E}((S^m \times S^n)_{\mathbb{Q}})$  if and only if  $m \neq n + 1$ .

(4) Since the fibrations are non-trivial, the models of the total spaces are uniquely determined as  $M(E) \cong (\Lambda(x, y, v_1), d)$  with  $|x| = m$ ,  $|y| = n$ ,  $dx = dy = 0$  and  $dv_1 = xy$  and  $M(E') \cong (\Lambda(x, y, v_1, v_2), d)$  with  $dx = dy = 0$ ,  $dv_1 = xy$ ,  $dv_2 = xv_1$ . Then  $\psi_{p'_{\mathbb{Q}}} \circ \psi_{p_{\mathbb{Q}}} : \mathcal{E}(\Lambda(x, y), 0) \cong \mathcal{E}(\Lambda(x, y, v_1), d) \cong \mathcal{E}(\Lambda(x, y, v_1, v_2), d)$  from degree reasons.

(5) Recall the rigid model of Arkowitz and Lupton [4]: Let

$$M = (\Lambda(x_1, x_2, y_1, y_2, y_3, z), d)$$

with given by  $|x_1| = 8$ ,  $|x_2| = 10$ ,  $|y_1| = 33$ ,  $|y_2| = 35$ ,  $|y_3| = 37$  and  $|z| = 119$ ,

$$dx_1 = dx_2 = 0, \quad dy_1 = x_1^3 x_2, \quad dy_2 = x_1^2 x_2^2, \quad dy_3 = x_1 x_2^3 \quad \text{and}$$

$$dz = \alpha := x_1^4 (y_1 x_2 - x_1 y_2) (y_2 x_2 - x_1 y_3) + x_1^{15} + x_2^{12}.$$

(Note that the degrees of elements are determined by the differential  $d$ .) Then  $\mathcal{E}(M) = \{id_M\}$  [4, Example 5.1]. Now define a Hirsch extension [12] by  $M$  as

$$(\Lambda(v, w), 0) \rightarrow (\Lambda(v, w) \otimes M, d') =: M'$$

where  $|v|$  and  $|w|$  are suitable odd integers with  $|v| \neq |w|$ ,  $|v| + |w| = 120$  and  $d'v = d'w = 0$ ,  $d'z = \alpha + vw$  ( $d' = d$  for the other elements). For example, put  $|v| = 53$  and  $|w| = 67$ . Then any DGA-automorphism  $h$  of  $M'$  is DGA-homotopic to a map with no unipotent part. Indeed, let

$$h(z) = z + f_1 v + f_2 w, \quad h(v) = av, \quad h(w) = bw + f_3 v$$

where  $a, b \in \mathbb{Q}^*$  with  $ab = 1$  and  $f_1, f_2, f_3 \in (x_1, x_2)$ , which is the ideal generated by  $x_1, x_2$ . Then, since  $|f_1| = 66$ ,  $|f_2| = 52$  and  $|f_3| = 14$ , we have

$$f_1 = k_1 x_1^2 x_2^5 + l_1 x_1^7 x_2, \quad f_2 = k_2 x_1^4 x_2^2, \quad f_3 = 0$$

for  $k_1, l_1, k_2 \in \mathbb{Q}$ . Thus  $f_1$  and  $f_2$  are  $d'$ -exact cocycles. Therefore  $h$  is DGA-homotopic to the map with  $f_1 = f_2 (= f_3) = 0$  [4],[12]. Hence any element  $h \in \mathcal{E}(M')$  is determined by  $h(v) = av$  and  $h(w) = bw$  for  $a, b \in \mathbb{Q}^*$  such that  $ab = 1$  (since  $h|_{M \sim id_M}$ ). Thus we obtain

$$\mathcal{E}(M') = \{(a, b) \in \mathbb{Q}^* \times \mathbb{Q}^* \mid ab = 1\} \cong \mathbb{Q}^*.$$

Therefore the DGA-surjections  $(\Lambda v, 0) \xleftarrow{f^*} M' \xrightarrow{g^*} (\Lambda w, 0)$  (spherically injective maps  $f : S_{\mathbb{Q}}^{[v]} \rightarrow ||M'||$  and  $g : S_{\mathbb{Q}}^{[w]} \rightarrow ||M'||$ , which are their geometric realizations) with  $f^*(M) = g^*(M) = 0$ ,  $f^*(v) = v$ ,  $g^*(w) = w$  and  $f^*(w) = g^*(v) = 0$  induce

$$(\mathbb{Q}^* \cong) \mathcal{E}(S_{\mathbb{Q}}^{[v]}) = \mathcal{E}(\Lambda v, 0) \underset{\phi_f}{\cong} \mathcal{E}(M') \underset{\psi_g}{\cong} \mathcal{E}(\Lambda w, 0) = \mathcal{E}(S_{\mathbb{Q}}^{[w]})$$

with  $\psi_g \phi_f(a) = a^{-1}$  for  $a \in \mathbb{Q}^*$ . Thus we have  $S_{\mathbb{Q}}^{[v]} \underset{\mathcal{E}}{\sim} S_{\mathbb{Q}}^{[w]}$ .  $\square$

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